

On pairs of geometric foliations on a cuspidal edge

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We study the topological configurations of the lines of principal curvature, the asymptotic and characteristic curves on a cuspidal edge, in the domain of a parametrization of this surface as well as on the surface itself. Such configurations are determined by the 3-jets of a parametrization of the surface.

1 Introduction and preliminaries about cuspidal edges

A singular point x of a map $f : (\mathbf{R}^2, x) \rightarrow (\mathbf{R}^3, 0)$ is called a *cuspidal edge* if the map-germ f at x is \mathcal{A} -equivalent to $(u, v) \mapsto (u, v^2, v^3)$ at 0. (Two map-germs $f_1, f_2 : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^m, 0)$ are \mathcal{A} -equivalent if there exist diffeomorphisms $S : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and $T : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^m, 0)$ such that $f_2 \circ S = T \circ f_1$.) If the singular point x of f is a cuspidal edge, then f at x is a front in the sense of [1] (see also [21]), and furthermore, they are one of two types of generic singularities of fronts (the other one is a *swallowtail* which is a singular point u of f satisfying that f at u is \mathcal{A} -equivalent to $(u, v) \mapsto (u, u^2v + 3u^4, 2uv + 4u^3)$ at 0).

It is shown in [22] that a cuspidal edge can locally be parametrized after smooth changes of coordinates in the source and isometries in the target by

$$f(u, v) = \left(u, a_1(u) + \frac{v^2}{2}, b_2(u) + v^2b_3(u) + v^3b_4(u, v) \right), \quad (1.1)$$

where $a_1(u), b_2(u), b_3(u), b_4(u, v)$ are C^∞ -functions satisfying $a_1(0) = a_1'(0) = b_2(0) = b_2'(0) = b_3(0) = 0$, $b_2''(0) > 0$, $b_4(0, 0) \neq 0$. Writing $a_1(u) = a_{20}u^2/2 + a_{30}u^3/6 + u^4h_1(u)$, $b_2(u) = b_{20}u^2/2 + b_{30}u^3/6 + u^4h_2(u)$, $b_3(u) = b_{12}u/2 + u^2h_3(u)$, $b_4(u, v) = b_{03}/6 + uh_4(u) + vh_5(u, v)$, we have

$$f(u, v) = \left(u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{v^2}{2}, \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3 \right) + h(u, v), \quad (1.2)$$

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where $b_{03} \neq 0$, $b_{20} \geq 0$ and

$$h(u, v) = (0, u^4 h_1(u), u^4 h_2(u) + u^2 v^2 h_3(u) + uv^3 h_4(u) + v^4 h_5(u, v)),$$

with $h_1(u), h_2(u), h_3(u), h_4(u), h_5(u, v)$ smooth functions. Several differential geometric invariants of cuspidal edges are investigated ([20, 22, 23, 25, 27, 28]), and coefficients of (1.2) are such invariants. According to [22], it is known that a_{20} coincides with the singular curvature κ_s , b_{20} coincides with the limiting normal curvature κ_ν , b_{03} coincides with the cuspidal curvature κ_c and b_{12} coincides with the cusp-directional torsion κ_t at the origin. The singular curvature is the geodesic curvature of the singular set with sign, and the limiting normal curvature is the normal curvature of the singular set, and they relates to the shape of cuspidal edge (see [28]). The cuspidal curvature measures the wideness of the cusp, and the cusp-directional torsion measures the rotating ratio of the cusp along the singular set (see [22, 23]).

On the other hand, let $U \subset \mathbf{R}^2$ be an open subset and (u, v) a coordinate system on U . Let

$$\omega = a(u, v)dv^2 + 2b(u, v)dudv + c(u, v)du^2 \quad (1.3)$$

be a 2-tensor on U , where a, b, c are smooth functions, called the *coefficients* of ω . We call $\omega = 0$ a *binary differential equation (BDE) corresponding to ω* . If $b^2 - ac > 0$ at $x \in U$, then $\omega(x) = 0$ defines two directions in $T_x U$, and integral curves of these directions for two smooth and transverse foliations, called *foliations with respect to ω* . If $b^2 - ac = 0$ at $x \in U$, then generically $\omega(x) = 0$ defines a single direction, and the integral curves form in general a family of cusps. Thus we are mainly interested in behavior of integral curves of a BDE near a point where $b^2 - ac$ vanishes. We call *discriminant* of a BDE the set where $b^2 - ac = 0$. If the single direction is transverse to the discriminant, then the BDE is equivalent to $dv^2 + udu^2 = 0$ ([7, 8]). The normal form for the stable cases when the single direction is tangent to the discriminant is obtained in [9, 10]. Topological classifications of generic families of BDEs are obtained in [3, 5, 6, 26, 29, 32]. On the other hand, BDEs as geometric foliations on surfaces in three space is studied in [2, 12, 15, 16, 17, 29]. See [11, 18, 19] for other approaches for geometric foliations.

In this paper, following [12, 13, 15, 26, 29], we stick to special BDEs from differential geometry of surface in \mathbf{R}^3 . There are three fundamental BDEs on a regular surface in \mathbf{R}^3 . Let $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$ be a regular surface with a unit normal vector field ν . Let ω_{lc} , ω_{as} and ω_{ch} be 2-tensors defined by

$$\begin{aligned} \omega_{lc} &= (EM - FL)dv^2 + (EN - GL)dudv + (FN - GM)du^2, \\ \omega_{as} &= N dv^2 + 2M dudv + L du^2, \\ \omega_{ch} &= (2M(GM - FN) - N(GL - EN)) dv^2 \\ &\quad + 2(M(GL + EN) - 2FLN) dudv \\ &\quad + (L(GL - EN) - 2M(FL - EM)) du^2, \end{aligned}$$

where (u, v) is a coordinate system on $(\mathbf{R}^2, 0)$, and

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\ \langle f_v, f_u \rangle & \langle f_v, f_v \rangle \end{pmatrix}, \quad \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} \langle f_{uu}, \nu \rangle & \langle f_{uv}, \nu \rangle \\ \langle f_{uv}, \nu \rangle & \langle f_{vv}, \nu \rangle \end{pmatrix},$$

where $\langle \cdot, \cdot \rangle$ stands for the standard inner product of \mathbf{R}^3 . Each integral curve of the foliations with respect to ω_{lc} is called a *line of curvature*, each integral curve with respect to ω_{as} is called an *asymptotic curve* and each integral curve with respect to ω_{ch} is called a *characteristic curve* or *harmonic mean curvature curve*. Asymptotic curves appear only on a domain where the Gaussian curvature K of f is non-negative, and characteristic curves appear only on a domain where the Gaussian curvature K of f is non-positive. Since $\omega_{ch} = 0$ can be deformed as

$$\begin{aligned} & (NH - GK) dv^2 + 2(MH - FK) dudv + (LH - EK) du^2 = 0 \\ \Leftrightarrow & \frac{N dv^2 + 2M dudv + L du^2}{G dv^2 + 2F dudv + E du^2} = \frac{K}{H} \left(= \frac{2}{\kappa_1^{-1} + \kappa_2^{-1}} \right), \end{aligned}$$

where K is the Gaussian curvature, H is the mean curvature, and κ_1, κ_2 are the principal curvatures of f , we see that along the characteristic curve, the normal curvature of it is equal to the harmonic mean of the principal curvatures (see [13], for example).

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2 Preliminaries on BDEs

In this section, following [5, 6], we introduce a method to study the configurations of the solution curves of a BDE. Let $\omega(u, v)$ be the 2-tensor on $(U; (u, v)) \subset \mathbf{R}^2$ as in (1.3). If $(a, b, c) \neq (0, 0, 0)$ at $x \in U$, then ω is called of *Type 1* at x , and if $(a, b, c) = (0, 0, 0)$ at $x \in U$, then ω is called of *Type 2* at x . If ω is of Type 2 at x , then $\delta = b^2 - ac$ has a critical point at x . Since we are interested in local behavior of ω , we set $x = (0, 0)$. If ω is of Type 1, then ω defines a single direction at points on Δ , and if it is of Type 2, then all directions in the plane are solutions of $\omega = 0$ at that point. Moreover, if ω is of Type 2 at x , then Δ is not a smooth curve. We are interested in the configurations of the foliations of $\omega = 0$. We define the following equivalence.

Definition 2.1. Two binary differential equations $\omega_1 = 0$ and $\omega_2 = 0$ are *equivalent* if there exist a diffeomorphism germ $\Phi : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ and a non-zero function $\rho : (\mathbf{R}^2, 0) \rightarrow \mathbf{R}$ such that $\rho(\Phi^*\omega_1) = \omega_2$ holds. If Φ is a homeomorphism such that Φ takes the integral curves of ω_1 to those of ω_2 , they are called *topologically equivalent*.

If two binary differential equations are equivalent then the configurations of their foliations can be regarded the same. To obtain the topological configurations, we use the following method developed in [5, 6, 12, 15, 29, 30]. We separate our consideration into the following three cases:

- Case 1: $(a(0), b(0), c(0)) \neq (0, 0, 0)$ and $\delta(0) \neq 0$ (Type 1).
- Case 2: $(a(0), b(0), c(0)) \neq (0, 0, 0)$ and $\delta(0) = 0$ (Type 1).
- Case 3: $(a(0), b(0), c(0)) = (0, 0, 0)$ (Type 2).

Consider the associated surface to ω

$$\mathcal{M} = \{(u, v, [\alpha, \beta]) \in (\mathbf{R}^2, 0) \times \mathbf{R}P^1 \mid a\beta^2 + 2b\alpha\beta + c\alpha^2 = 0\}.$$

Then \mathcal{M} is a smooth manifold if $b^2 - ac \neq 0$, or if $a_u + 2b_up + c_up^2 = 0$ and $a_v + 2b_vp + c_vp^2 = 0$ do not have any common root. The second condition is equivalent to

$$\det \begin{pmatrix} a_u & 2b_u & c_u & 0 \\ 0 & a_u & 2b_u & c_u \\ a_v & 2b_v & c_v & 0 \\ 0 & a_v & 2b_v & c_v \end{pmatrix} (0) \neq 0. \quad (2.1)$$

Consider the projection $\pi : \mathcal{M} \rightarrow \mathbf{R}^2$, $\pi(u, v, [\alpha : \beta]) = (u, v)$. Then $\pi^{-1}(u, v)$ consists of two points if $b^2 - ac > 0$, and is empty if $b^2 - ac < 0$. Let us set $p = \beta/\alpha$ (we need to consider the case $q = \alpha/\beta$ for some cases) and $\mathcal{F}(u, v, p) = ap^2 + 2bp + c$. If $\mathcal{F}_p(0) \neq 0$, then π is a local diffeomorphism, and if $\mathcal{F}_p(0) = 0$ and $\mathcal{F}_{pp}(0) \neq 0$ hold, then π is a fold (a map-germ $h : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ is a *fold* if h is \mathcal{A} -equivalent to $(u, v) \mapsto (u, v^2)$). Let us consider the vector field

$$\xi(u, v, p) = p\mathcal{F}_p(u, v, p)\partial_u + \mathcal{F}_p(u, v, p)\partial_v - (p\mathcal{F}_u(u, v, p) + \mathcal{F}_v(u, v, p))\partial p.$$

Then ξ is tangent to \mathcal{M} , and the projections $d\pi(\xi_1), d\pi(\xi_2)$ satisfy

$$\omega(d\pi(\xi_i), d\pi(\xi_i)) = 0 \quad (i = 1, 2),$$

where $\xi_i = \xi(u, v, p_i)$ and $\pi^{-1}(u, v) = \{(u, v, p_1), (u, v, p_2)\}$, or $\pi^{-1}(u, v) = \{(u, v, p_1)\}$. To study geometric the foliations of ω , we use ξ on \mathcal{M} .

2.1 Case 1

We assume that $b^2 - ac > 0$ at 0. Then one can easily see that the BDE (1.3) is equivalent to a BDE $a dv^2 + 2b dudv + c du^2 = 0$ which satisfies $(a(0, 0), b(0, 0), c(0, 0)) = (1, 0, -1)$. Furthermore, it holds that for any $k > 0$, a BDE $a dv^2 + 2b dudv + c du^2 = 0$ which satisfies $(a(0, 0), b(0, 0), c(0, 0)) = (1, 0, -1)$ is equivalent to a BDE $a dv^2 + 2b dudv +$

$c du^2 = 0$ whose k -jet of (a, b, c) at $(0, 0)$ is $(1, 0, -1)$, moreover, it is equivalent to a BDE

$$\omega_{reg} = dv^2 - du^2 = 0 \quad (2.2)$$

([6, Proposition 4.4]). Therefore the configuration is a pair of transverse smooth foliations.

2.2 Case 2

We consider the case 2, namely $(a(0), b(0), c(0)) \neq (0, 0, 0)$ and $\delta(0) = 0$.

Lemma 2.2. *We assume that $(a(0), b(0), c(0)) \neq (0, 0, 0)$ and $\delta(0) = 0$. If ω as in (1.3) satisfies*

$$j^1(a, b, c)(0, 0) = (0, 0, \alpha_0) + (\alpha_1 v, \alpha_2 v, \alpha_3 u + \alpha_4 v),$$

then it is equivalent to a BDE $a dv^2 + 2b dudv + c du^2 = 0$ which satisfies

$$(j^1 a(0, 0), j^1 b(0, 0), j^1 c(0, 0)) = (1, 0, u)$$

when $\alpha_0 \alpha_1 \neq 0$.

Proof. We assume that $\alpha_0 \alpha_1 \neq 0$. Consider $u = V + \beta_1 U^2 + \beta_2 V^2$, $v = U$ and $R = 1 + \beta_3 U$, where $\beta_1, \beta_2, \beta_3 \in \mathbf{R}$. Then $R\omega(U, V)$ is given by

$$\begin{aligned} & (\alpha_1 U + O(2)) dV^2 + (2(\alpha_2 + 2\alpha_0 \beta_1)U + O(2)) dU dV \\ & + (\alpha_0 + (\alpha_4 + \alpha_0 \beta_3)U + (\alpha_3 + 4\alpha_0 \beta_2)V + O(2)) dU^2, \end{aligned}$$

where $O(2)$ stands for remainders of order 2. Setting $\beta_1 = -\alpha_2/(2\alpha_0)$, $\beta_2 = -\alpha_3/(4\alpha_0)$ and $\beta_3 = -\alpha_4/\alpha_0$, and re-scaling, we get the desired result. \square

Any BDE of the form (1.3) with $(j^1 a(0, 0), j^1 b(0, 0), j^1 c(0, 0)) = (1, 0, u)$ is smoothly equivalent to

$$\omega_{cusp} = dv^2 + u du^2 = 0 \quad (2.3)$$

([8], see also [6, Section 4.2], [30, Proposition 3.3-2]). The solutions form a family of cusps.

2.3 Case 3

We assume $(a(0), b(0), c(0)) = (0, 0, 0)$. In this case, δ has a critical point at 0. We assume \mathcal{M} is a smooth manifold. Since $\mathcal{F}_p(0, 0, p) = 0$ holds, ξ has a zero at $(0, 0, p)$ if and only if $p\mathcal{F}_u(0, 0, p) + \mathcal{F}_v(0, 0, p) = 0$. This is a cubic equation for p . Set

$$\varphi_\omega(p) = \varphi(p) = p\mathcal{F}_u(0, 0, p) + \mathcal{F}_v(0, 0, p). \quad (2.4)$$

Let $D_\omega = D$ denotes the discriminant of this equation. If $D > 0$ then $\varphi(p) = 0$ has three distinct real roots, and if $D < 0$ then $\varphi(p) = 0$ has one real and two distinct imaginary roots.

When $D > 0$, let p_1, p_2, p_3 be the solutions of $\varphi(p) = 0$ and $p_1 < p_2 < p_3$. When $D < 0$, let p_1 be the solution of $\varphi(p) = 0$. If $\varphi(0) = \mathcal{F}_v(0, 0, 0) = a_v(0) \neq 0$ then $p_i \neq 0$ ($i = 1, 2, 3$ or $i = 1$) holds. We need to understand the singularity of ξ near p_i . If $\mathcal{F}_u(0, 0, p_i) \neq 0$, then \mathcal{M} is parameterized by v as $(u(v, p), v, p)$ near $(0, 0, p_i)$. We denote the linear part of ξ by

$$j^1\xi(0, 0, p_i) = (\xi_{11}v + \xi_{12}(p - p_i))\partial v + (\xi_{21}v + \xi_{22}(p - p_i))\partial p.$$

Also we remark that since $\mathcal{F}(u(v, p), v, p) \equiv 0$, it holds that $\mathcal{F}_u u_v + \mathcal{F}_v \equiv 0$, where \equiv means that the equality holds identically. On the other hand, p_i is a solution of $\varphi(p) = 0$, and $\varphi(p) = p\mathcal{F}_u + \mathcal{F}_v$, we have

$$u_v(0, p_i) = p_i.$$

Furthermore, $\mathcal{F}_p = 0$ at $(u, v) = (0, 0)$, it holds that $u_p = 0$ at $(v, p) = (0, p_i)$. We have

$$\begin{aligned} \xi_{11} &= \mathcal{F}_{up}u_v + \mathcal{F}_{pv} \\ \xi_{12} &= \mathcal{F}_{up}u_p + \mathcal{F}_{pp} \\ &= 0 \quad (\text{at } (0, 0, p_i)) \\ \xi_{22} &= -(\mathcal{F}_u + p\mathcal{F}_{uu}u_p + p\mathcal{F}_{up} + \mathcal{F}_{uv}u_p + \mathcal{F}_{vp}) \end{aligned}$$

Thus the eigenvalues of the linear part of ξ are

$$\alpha(p_i) = \mathcal{F}_{up}u_v + \mathcal{F}_{pv} \text{ and } -\varphi'(p_i) = -(\mathcal{F}_u + p\mathcal{F}_{uu}u_p + p\mathcal{F}_{up} + \mathcal{F}_{uv}u_p + \mathcal{F}_{vp}). \quad (2.5)$$

The configuration of the integral curves of ω is determined by these information. The following theorem is known. Let $\det \text{Hess } \delta(0, 0)$ be the determinant of the Hesse matrix of $\delta(u, v)$ at $(0, 0)$.

Theorem 2.3. [5, Theorem 4.1] *Let ω be a 2-tensor as in (1.3) and satisfies $(a, b, c)(0) = (0, 0, 0)$, $\det \text{Hess } \delta(0, 0) < 0$, $D \neq 0$ and $\varphi(p)$ and $\alpha(p)$ do not have any common roots. Then the BDE $\omega = 0$ is topologically equivalent to one of the following BDEs:*

- The case $D > 0$: (Then $\varphi(p) = 0$ has 3 roots p_1, p_2, p_3 .)
 - $\omega_{3s} = vdv^2 + 2ududv + vdu^2 = 0$ (3 saddles) when $-\varphi'(p_i)\alpha(p_i)$ are negative for all $i = 1, 2, 3$.
 - $\omega_{2s1n} = vdv^2 + 2(v - u)dudv + vdu^2 = 0$ (2 saddles + 1 node) when $-\varphi'(p_i)\alpha(p_i)$ are two negative and one positive for $i = 1, 2, 3$.
 - $\omega_{1s2n} = 3vdv^2 - 4ududv + 3vdu^2 = 0$ (1 saddle + 2 nodes) when $-\varphi'(p_i)\alpha(p_i)$ are one negative and two positive for $i = 1, 2, 3$.

- The case $D < 0$: (Then $\varphi(p) = 0$ has 1 root p_1 .)
 - $\omega_{1s} = vdv^2 + 2ududv + vdu^2 = 0$ (1 saddle) when $-\varphi'(p_1)\alpha(p_1)$ is positive.
 - $\omega_{1n} = 2v dv^2 - ududv + 2vdu^2 = 0$ (1 node) when $-\varphi'(p_1)\alpha(p_1)$ is negative.

Note that in the case of $D > 0$, all $-\varphi'(p_i)\alpha(p_i)$ $i = 1, 2, 3$ cannot be positive, see [5]. The integral curves of the above BDEs are in Figures 1, 2, 3 which are taken from [5].

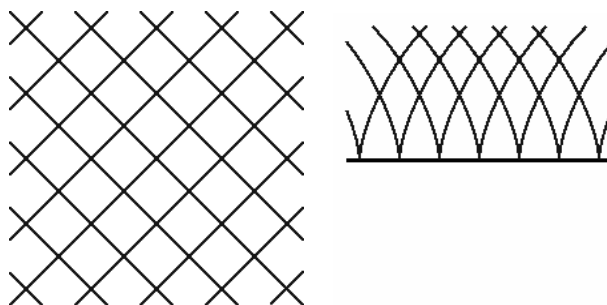


Figure 1: Integral curves of ω_{reg} and ω_{cusp} .

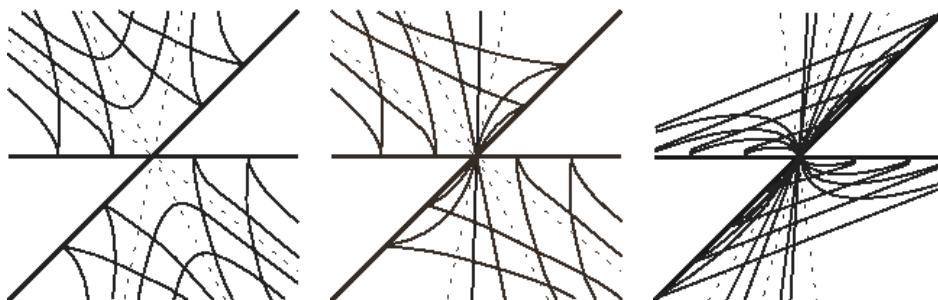


Figure 2: Integral curves of ω_{3s} , ω_{2s1n} and ω_{1s2n} . Here and in the rest of the paper, the dashed curves are separatrices, i.e., they are integral curves that separate distinct topological sectors.

3 Geometric binary differential equations on a cuspidal edge

Let $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$ be a parametrization of a cuspidal edge. We take f as in (1.2). Then the coefficients of the first fundamental form of the cuspidal edge with

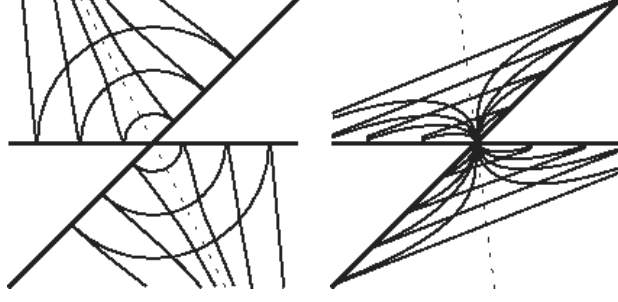


Figure 3: Integral curves of ω_{1s} and ω_{1n} .

respect to f are

$$\begin{aligned}
E &= 1 + (a_{20}^2 + b_{20}^2)u^2 + (a_{20}a_{30} + b_{20}b_{30})u^3 + b_{12}b_{20}uv^2 \\
&\quad + \frac{1}{4}(a_{30}^2 + b_{30}^2 + 2a_{20}p(0) + 2b_{20}q_1(0))u^4 \\
&\quad + \frac{1}{2}(b_{12}b_{30} + 8b_{20}q_2(0))u^2v^2 + 2b_{20}q_3(0)uv^3 \\
&\quad + \frac{1}{4}b_{12}^2v^4 + O(5),
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
F &= a_{20}uv + \left(\frac{1}{2}a_{30} + b_{12}b_{20}\right)u^2v + \frac{1}{2}b_{03}b_{20}uv^2 \\
&\quad + \frac{1}{4}b_{03}b_{12}v^4 + \left(\frac{1}{2}b_{12}^2 + 4b_{20}q_4(0,0)\right)uv^3 \\
&\quad + \frac{1}{4}(b_{03}b_{30} + 12b_{20}q_3(0))u^2v^2 \\
&\quad + \left(\frac{1}{2}b_{12}b_{30} + 4p(0) + 2b_{20}q_2(0)\right)u^3v + O(5),
\end{aligned} \tag{3.2}$$

$$G = v^2 + \frac{1}{4}b_{03}^2v^4 + b_{03}b_{12}uv^3 + b_{12}^2u^2v^2 + O(5), \tag{3.3}$$

where $O(n)$ stands for remainders of order n ($n = 1, 2, \dots$). Since f is as in (1.2), we see $f_u \times (f_v/v) \neq 0$. We set $\nu_2 = f_u \times (f_v/v)$, and $L_2 = \langle f_{uu}, \nu_2 \rangle$, $M_2 = \langle f_{uv}, \nu_2 \rangle$, $N_2 = \langle f_{vv}, \nu_2 \rangle$. Then we have:

$$\begin{aligned}
L_2 &= b_{20} + (b_{30} - a_{20}b_{12})u - \frac{1}{2}a_{20}b_{03}v \\
&\quad - (a_{30}b_{12} - 12q_1(0) + 2a_{20}q_2(0))u^2 \\
&\quad - (a_{30}b_{03} + 6a_{20}q_3(0))uv \\
&\quad + (2q_2(0) - 4a_{20}q_4(0,0))v^2 + O(3) \\
M_2 &= b_{12}v + 4q_2(0)uv + 3q_3(0)v^2 + O(3) \\
N_2 &= \frac{1}{2}b_{03}v + 3q_3(0)uv + 8q_4(0,0)v^2 + O(3).
\end{aligned} \tag{3.4}$$

We have $L_2 = L|\nu_2|$, $M_2 = M|\nu_2|$, $N_2 = N|\nu_2|$. It should be remarked that there exist C^∞ -functions $\widetilde{F}, \widetilde{G}, \widetilde{N}, \widetilde{M}$ such that $G = v^2\widetilde{G}$, $F = v\widetilde{F}$, $N_2 = v\widetilde{N}$ and $M_2 = v\widetilde{M}$ holds. We set

$$\widetilde{E} = E, \quad \widetilde{F} = \frac{F}{v}, \quad \widetilde{G} = \frac{G}{v^2}, \quad \widetilde{L} = L_2, \quad \widetilde{M} = \frac{M_2}{v}, \quad \widetilde{N} = \frac{N_2}{v}. \quad (3.5)$$

3.1 Lines of principal curvature

In this subsection we consider the BDE $\omega_{lc} = 0$. Using (3.5), $\omega_{lc} = 0$ is equivalent to

$$v^2 \left(\widetilde{F}\widetilde{N} - v\widetilde{G}\widetilde{M} \right) dv^2 + v \left(\widetilde{E}\widetilde{N} - v^2\widetilde{G}\widetilde{L} \right) dudv + v \left(\widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L} \right) du^2 = 0.$$

To determine the topological configuration of $\omega_{lc} = 0$, we factor out v and consider $\widetilde{\omega}_{lc} = 0$, where

$$\widetilde{\omega}_{lc} = v \left(\widetilde{F}\widetilde{N} - v\widetilde{G}\widetilde{M} \right) dv^2 + \left(\widetilde{E}\widetilde{N} - v^2\widetilde{G}\widetilde{L} \right) dudv + \left(\widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L} \right) du^2.$$

We have the following proposition.

Proposition 3.1. *The BDE $\widetilde{\omega}_{lc} = 0$ is equivalent to the BDE $\omega_{reg} = 0$.*

This proposition implies that the lines of principal curvature of a cuspidal edge form a pair of smooth and transverse foliations in the domain of a parametrization.

Proof of Proposition 3.1. Set

$$(a, b, c) = \left(v\widetilde{F}\widetilde{N} - v^2\widetilde{G}\widetilde{M}, \frac{1}{2} \left(\widetilde{E}\widetilde{N} - v\widetilde{G}\widetilde{L} \right), \widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L} \right).$$

Then since $b(0) = \widetilde{E}(0)\widetilde{N}(0)$, $\widetilde{E}(0) = \langle f_u, f_u \rangle(0) \neq 0$ and $\widetilde{N}(0) = \langle f_{vv}, \nu_2 \rangle(0) = b_{03} \neq 0$ hold, $\widetilde{\omega}_{lc}$ is as in Case 1. Moreover, since $a(0) = 0$ and $b(0) \neq 0$, we see that $b^2 - ac > 0$ at 0. Hence $\widetilde{\omega}_{lc}$ is equivalent to $\omega_{reg} = 0$ (See Section 2.1). \square

The fact of the existence of the curvature line coordinate system at cuspidal edge is also shown in [24].

An example of picture of this configuration on the cuspidal edges is in Figures 4. Since one family of the integral curves are tangent to the null direction on singular curve, one family of the integral curves near singular curve form the $(2, 3)$ -cusps. A map-germ $(\mathbf{R}, 0) \rightarrow (\mathbf{R}^3, 0)$ is an $(2, 3)$ -cusp if it is \mathcal{A} -equivalent to $t \mapsto (t^2, t^3)$.

3.2 Asymptotic curves

In this subsection we consider $\omega_{as} = 0$, and it is equivalent to $\widetilde{\omega}_{as} = N_2 dv^2 + 2M_2 dudv + N_2 du^2 = 0$. Since $M_2(u, 0) = 0$, and $N_2(u, 0) = 0$, the singular set i.e., the cuspidal

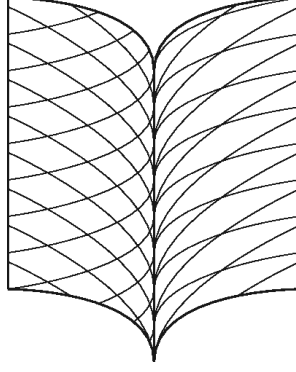


Figure 4: Integral curves of $\tilde{\omega}_{lc}$ on images of cuspidal edges.

edge curve is part of discriminant of $\tilde{\omega}_{as} = 0$, and, ∂_v is a solution to $\tilde{\omega}_{as} = 0$ on the singular set. By (3.4),

$$\begin{aligned}
\delta(u, v) = & -\frac{1}{2}b_{20}b_{03}v + \left(\frac{1}{4}a_{20}b_{03}^2 + b_{12}^2 - 8b_{20}q_4(0, 0)\right)v^2 \\
& + \frac{1}{2}b_{03}\left(a_{20}b_{12} - b_{30} - 2b_{20}q_3(0)\right)uv \\
& + \left(-b_{03}q_2(0) + 6b_{12}q_3(0) \right. \\
& \quad \left. + 6a_{20}b_{03}q_4(0, 0) - 7b_{20}(q_4)_v(0, 0)\right)v^3 \\
& + \left(\frac{1}{4}a_{30}b_{03}^2 + 3a_{20}b_{03}q_3(0) - 8b_{30}q_4(0, 0) \right. \\
& \quad \left. + 8b_{12}q_2(0) + 8a_{20}b_{12}q_4(0, 0)\right)uv^2 \\
& + \left(\frac{1}{2}a_{30}b_{03}b_{12} - 6b_{03}q_1(0) + a_{20}b_{03}q_2(0) \right. \\
& \quad \left. + 3a_{20}b_{12}q_3(0) - 3b_{30}q_3(0)\right)u^2v + O(4).
\end{aligned} \tag{3.6}$$

Thus if $b_{20} \neq 0$, then the BDE is in Case 2. In this case, by (3.4), $\omega_{as} = 0$ is equivalent to $\omega_{cusp} = 0$ (see Subsection 2.2). By [28, Corollary 3.6], $b_{20} \neq 0$ implies that the Gaussian curvature is unbounded and changes sign between the two sides of the cuspidal edge. This means that in this case, the singular set of cuspidal edge plays the same role as the parabolic curve on regular surfaces. Since $b_{20} \neq 0$, then the BDE is $\omega_{cusp} = 0$, this implies that the folded saddle, the folded node, the folded focus in Davydov's classification [9] (see also [30, Section 3.2]) does not appear not only cuspidal edges, but also all singularities written in the form (1.1) (for instance, cuspidal cross cap).

We assume now $b_{20} = 0$. Then the BDE is in Case 3. We have the following. If $a_{20}b_{12} - b_{30} \neq 0$, then δ is a Morse function near 0. We study the geometric foliation

near 0 as in Subsection 2.3. We consider

$$\mathcal{F}(u, v, p) = N_2 + 2M_2p + L_2p^2.$$

Then we have

$$\begin{aligned} (\mathcal{F}_u, \mathcal{F}_v, \mathcal{F}_p)(u, v, p) &= ((N_2)_u + 2(M_2)_up + (L_2)_up^2, \\ &\quad (N_2)_v + 2(M_2)_vp + (L_2)_vp^2, 2M_2 + 2L_2p), \\ (\mathcal{F}_u, \mathcal{F}_v, \mathcal{F}_p)(0, 0, p) &= ((L_2)_up^2, (N_2)_v + 2(M_2)_vp + (L_2)_vp^2, 2Lp) \\ &= \left((b_{30} - a_{20}b_{12})p^2, \frac{1}{2}(b_{03} + 4b_{12}p - a_{20}b_{03}p^2), 0 \right). \end{aligned}$$

In this case, the left hand side of (2.1) is $(b_{30} - a_{20}b_{12})^2b_{03}^2/4$. Thus if $b_{30} - a_{20}b_{12} \neq 0$ at 0, then $\mathcal{M} = \{\mathcal{F} = 0\}$ is a smooth manifold. We have $\varphi_{as} = \varphi_{\omega_{as}}$ and $D_{as} = D_{\omega_{as}}$ defined in Subsection 2.3 as follows

$$\begin{aligned} \varphi_{as}(p) &= (b_{30} - a_{20}b_{12})p^3 - \frac{1}{2}a_{20}b_{03}p^2 + 2b_{12}p + \frac{1}{2}b_{03}, \\ 4D_{as} &= a_{20}^3b_{03}^4 + 13a_{20}^2b_{03}^2b_{12}^2 - b_{30}(128b_{12}^3 + 27b_{03}^2b_{30}) \\ &\quad + 2a_{20}(64b_{12}^4 + 9b_{03}^2b_{12}b_{30}). \end{aligned}$$

Furthermore, $\alpha(p)$ is given by

$$\alpha(p) = 2(b_{30} - a_{20}b_{12})p^2 - a_{20}b_{03}p + 2b_{12}.$$

If p_i is a solution of $\varphi_{as}(p) = 0$ and $p\alpha(p) - 2\varphi_{as}(p) = -b_{03} - 2b_{12}p$ holds, then $\alpha(p_i) \neq 0$ if and only if $-b_{03} - 2b_{12}p_i \neq 0$. Assume that $b_{12} \neq 0$. Substituting $p = -b_{03}/(2b_{12})$ into $\varphi_{as}(p)$, we get

$$\varphi_{as}\left(\frac{-b_{03}}{2b_{12}}\right) = -\frac{1}{8b_{12}^3}b_{03}(4 + b_{03}^2b_{30}).$$

If $b_{12} = 0$, then $p\alpha(p) - 2\varphi_{as}(p) = -b_{03} \neq 0$. Since we assume that $b_{30} - a_{20}b_{12} \neq 0$, if $b_{12} = 0$ then $b_{30} \neq 0$. Thus we get $\alpha(p_i) \neq 0$ if and only if

$$4b_{12}^3 + b_{03}^2b_{30} \neq 0.$$

We can now use Theorem 2.3 to obtain the following result.

Proposition 3.2. *If $b_{20} \neq 0$, then ω_{as} is equivalent to $\omega_{cusp} = 0$.*

If $b_{20} = 0$, $b_{30} - a_{20}b_{12} \neq 0$, $D_{as} \neq 0$ and $4b_{12}^3 + b_{03}^2b_{30} \neq 0$, then ω_{as} is topologically equivalent to one of the following:

- The case $D_{as} > 0$: (Then $\varphi_{as}(p) = 0$ has 3 roots p_1, p_2, p_3)
 - $\omega_{3s}(-\varphi'_{as}(p_i)\alpha(p_i))$ are negative for all $i = 1, 2, 3$.
 - $\omega_{2s1n}(-\varphi'_{as}(p_i)\alpha(p_i))$ are two negative and one positive for $i = 1, 2, 3$.
 - $\omega_{1s2n}(-\varphi'_{as}(p_i)\alpha(p_i))$ are one negative and two positive for $i = 1, 2, 3$.

- The case $D_{as} < 0$: (Then $\varphi_{as}(p) = 0$ has 1 root p_1)

– $\omega_{1s} (-\varphi'_{as}(p_1)\alpha(p_1))$ is negative).

– $\omega_{1n} (-\varphi'_{as}(p_1)\alpha(p_1))$ is positive).

Remark 3.3. We observe that by the Proposition 3.2, b_{20} , $b_{30} - a_{20}b_{12}$ and $4b_{12}^3 + b_{03}^2b_{30}$ have geometric meanings. In fact, b_{20} is the limiting normal curvature and $b_{30} - a_{20}b_{12}$ coincides with the derivation of limiting normal curvature (see [23]). The invariant $4b_{12}^3 + b_{03}^2b_{30}$ is related to the singularities of parallel surfaces of the cuspidal edge (see [31, Lemma 3.1]).

3.3 Characteristic curves

We consider the BDE $\omega_{ch} = 0$. Using (3.5), we show that $\omega_{ch} = 0$ is equivalent to $v(a dv^2 + 2b dudv + c du^2) = 0$, where

$$\begin{aligned} a &= v(\tilde{E}\tilde{N}^2 + (-\tilde{G}\tilde{L}\tilde{N} - 2\tilde{F}\tilde{M}\tilde{N})v + 2\tilde{G}\tilde{M}^2v^2) \\ b &= v(-2\tilde{F}\tilde{L}\tilde{N} + \tilde{E}\tilde{M}\tilde{N} + \tilde{G}\tilde{L}\tilde{M}v) \\ c &= -\tilde{E}\tilde{L}\tilde{N} + (\tilde{G}\tilde{L}^2 - 2\tilde{F}\tilde{L}\tilde{M} + 2\tilde{E}\tilde{M}^2)v. \end{aligned}$$

We factor out v , so $\omega_{ch} = 0$ is topologically equivalent to $\tilde{\omega}_{ch} = a dv^2 + 2b dudv + c du^2 = 0$. Since $a(u, 0) = 0$, and $b(u, 0) = 0$, the singular set is a part of discriminant of $\tilde{\omega}_{ch} = 0$, and ∂_v is a solution to $\tilde{\omega}_{ch} = 0$ on the singular set. The function $\delta = b^2 - ac$ is given by

$$\begin{aligned} v &\left[\tilde{E}^2\tilde{L}\tilde{N}^3 + (4\tilde{F}^2\tilde{L}^2 - 2\tilde{E}\tilde{G}\tilde{L}^2 - 4\tilde{E}\tilde{F}\tilde{L}\tilde{M} - 2\tilde{E}^2\tilde{M}^2)\tilde{N}^2v \right. \\ &\quad + (-\tilde{G}^2\tilde{L}^3 - 4\tilde{F}\tilde{G}\tilde{L}^2\tilde{M} - 4\tilde{F}^2\tilde{L}\tilde{M}^2 + 6\tilde{E}\tilde{G}\tilde{L}\tilde{M}^2 + 4\tilde{E}\tilde{F}\tilde{M}^3)\tilde{N}v^2 \\ &\quad \left. + (\tilde{G}\tilde{L}^2 - 4\tilde{F}\tilde{L}\tilde{M} + 4\tilde{E}\tilde{M}^2)\tilde{G}\tilde{M}^2v^3 \right]. \end{aligned}$$

When f is taken as in (1.2), we have

$$\begin{aligned} a &= \frac{1}{4}b_{03}^2v + O(2), \\ b &= \frac{1}{2}b_{12}b_{03}v + O(2), \\ c &= -\frac{1}{2}b_{20}b_{03} + \frac{1}{2}(a_{20}b_{12}b_{03} - b_{30}b_{03} - 6b_{20}q_3(0))u \\ &\quad + \frac{1}{4}[a_{20}b_{03}^2 + 8b_{12}^2 + 4b_{20}(b_{20} - 8q_4(0, 0))]v + O(2), \\ \delta &= \frac{1}{8}b_{03}^3b_{20}v + \frac{1}{8}b_{03}^2(-a_{20}b_{03}b_{12} + b_{03}b_{30} + 18b_{20}q_3(0))uv \\ &\quad - \frac{1}{16}b_{03}^2\left[a_{20}b_{03}^2 + 4\left[b_{12}^2 + 2b_{20}(b_{20} - 12q_4(0, 0))\right]\right]v^2 + O(3). \end{aligned}$$

Since $b_{03} \neq 0$, if $b_{20} \neq 0$, then $\tilde{\omega}_{ch}$ is as in Case 2, and if $b_{20} = 0$, then it is as in Case 3. In the following, we divide our consideration into these two cases.

The case $b_{20} \neq 0$:

By the argument in Subsection 2.2, $\tilde{\omega}_{ch} = 0$ is equivalent to $\omega_{cusp} = 0$. In this case, by (3.4), $\omega_{as} = 0$ is equivalent to $\omega_{cusp} = 0$ (see Subsection 2.2). Like as the case of $\omega_{as} = 0$, the singular set of cuspidal edge plays the same role as the parabolic curve on regular surfaces. Moreover, the folded saddle, the folded node, the folded focus do not appear.

The case $b_{20} = 0$:

The left hand side of (2.1) is $b_{03}^6(a_{20}b_{12} - b_{30})^2/64$. Thus \mathcal{M} is a smooth manifold if $a_{20}b_{12} - b_{30} \neq 0$ at 0. Furthermore, δ is of Morse type if and only if $a_{20}b_{12} - b_{30} \neq 0$. This is exactly the same conditions as the case of asymptotic curves. We assume that $a_{20}b_{12} - b_{30} \neq 0$. We need to consider $\mathcal{F}(u, v, p) = a + 2bp + cp^2$ and $\varphi_{ch}(p) = (p\mathcal{F}_u + \mathcal{F}_v)(0, 0, p)$. We have

$$4\varphi_{ch}(p) = b_{03}^2 + 4b_{03}b_{12}p + (a_{20}b_{03}^2 + 8b_{12}^2)p^2 + (2a_{20}b_{03}b_{12} - 2b_{03}b_{30})p^3, \quad (3.7)$$

and the discriminant $D_{ch} = D_{\omega_{ch}}$ of the cubic φ_{ch} is given by

$$D_{ch} = -\frac{b_{03}^2}{64} \left(a_{20}^3 b_{03}^6 + 11a_{20}^2 b_{03}^4 b_{12}^2 - 2a_{20}(16b_{03}^2 b_{12}^4 + 9b_{03}^4 b_{12} b_{30}) + 256b_{12}^6 + 160b_{03}^2 b_{12}^3 b_{30} + 27b_{03}^4 b_{30}^2 \right). \quad (3.8)$$

Furthermore, 4α is given by

$$4\alpha(p) = 2b_{03}(a_{20}b_{12} - b_{30})p^2 + (a_{20}b_{03}^2 + 8b_{12}^2)p + 2b_{12}b_{03}.$$

Then we have

$$4\varphi'_{ch}(p)\alpha(p) = 4b_{03}^2 b_{12}^2 + 4b_{03}b_{12}(a_{20}b_{03}^2 + 8b_{12}^2)p + (a_{20}^2 b_{03}^4 + 26a_{20}b_{03}^2 b_{12}^2 + 64b_{12}^4 - 10b_{03}^2 b_{12} b_{30})p^2 + 5b_{03}(a_{20}b_{03}^2 + 8b_{12}^2)(a_{20}b_{12} - b_{30})p^3 + 6b_{03}^2(-a_{20}b_{12} + b_{30})^2 p^4, \quad (3.9)$$

and the condition that $\varphi_{ch}(p)$ and $\alpha(p)$ do not have any common roots is given by $4b_{12}^3 + b_{03}^2 b_{30} \neq 0$. We summarize the above discussion in the following proposition.

Proposition 3.4. *If $b_{20} \neq 0$, then ω_{ch} is equivalent to $\omega_{cusp} = 0$.*

If $b_{20} = 0$, $b_{30} - a_{20}b_{12} \neq 0$, $D_{ch} \neq 0$ and $4b_{12}^3 + b_{03}^2 b_{30} \neq 0$, then ω_{ch} is topologically equivalent to one of the following:

- *The case $D_{ch} > 0$: (Then $\varphi_{ch}(p) = 0$ has 3 roots p_1, p_2, p_3)*
 - $\omega_{3s}(-\varphi'_{ch}(p_i)\alpha(p_i))$ are negative for all $i = 1, 2, 3$.

- $\omega_{2s1n} (-\varphi'_{ch}(p_i)\alpha(p_i))$ are two negative and one positive for $i = 1, 2, 3$).
- $\omega_{1s2n} (-\varphi'_{ch}(p_i)\alpha(p_i))$ are one negative and two positive for $i = 1, 2, 3$).
- The case $D_{ch} < 0$: (Then $\varphi_{ch}(p) = 0$ has 1 root p_1)
 - $\omega_{1s} (-\varphi'_{ch}(p_1)\alpha(p_1))$ is negative).
 - $\omega_{1n} (-\varphi'_{ch}(p_1)\alpha(p_1))$ is positive).

Remark 3.5. We remark that if $b_{12} = 0$, then $D_{as} = D_{ch}$. Namely, the configurations of foliations with respect to ω_{as} and ω_{ch} are of the same type.

Examples of pictures of these configurations on the cuspidal edges are in Figures 5, 6 and 7. Since the integral curves emanate from singular curve along the null direction, integral curves near the singular curve do not form the (2,3)-cusp but form the (3,4)-cusps (see Appendix A).

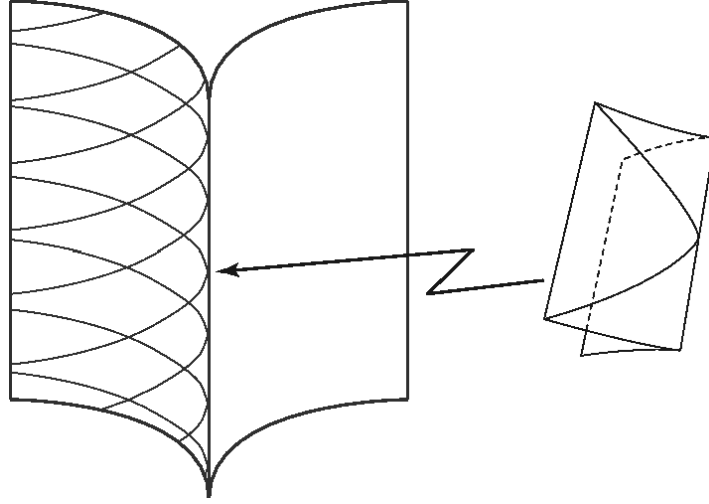


Figure 5: Integral curves of ω_{cusp} on images of cuspidal edges.

4 Generic foliations

In Propositions 3.2 and 3.4, all the conditions are written in terms of the 3-jet of (1.2). We can state a genericity result for cuspidal edge. By (1.2), we identify the set of jets of parametrization of cuspidal edges with $(0, 0)$

$$\mathcal{C}_k = \left\{ (j^k a_1(0), j^k b_2(0), j^k b_3(0), j^k b_4(0, 0)) \in J^k(1, 1)^3 \times J^k(2, 1) \mid \right. \\ \left. a_1(0) = a'_1(0) = b_2(0) = b'_2(0) = b_3(0) = 0, b'_2(0) > 0, b_4(0, 0) \neq 0 \right\},$$

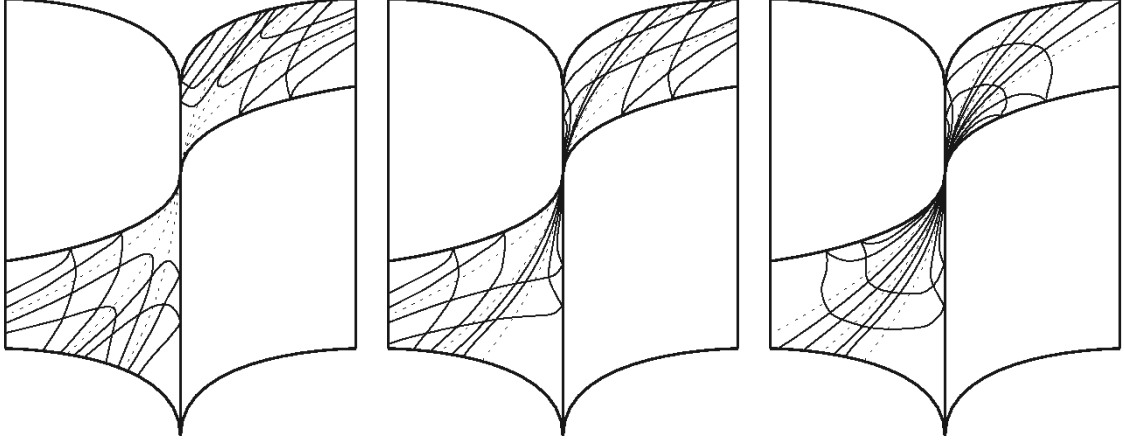


Figure 6: Integral curves of ω_{3s} , ω_{2s1n} and ω_{1s2n} on images of cuspidal edges.

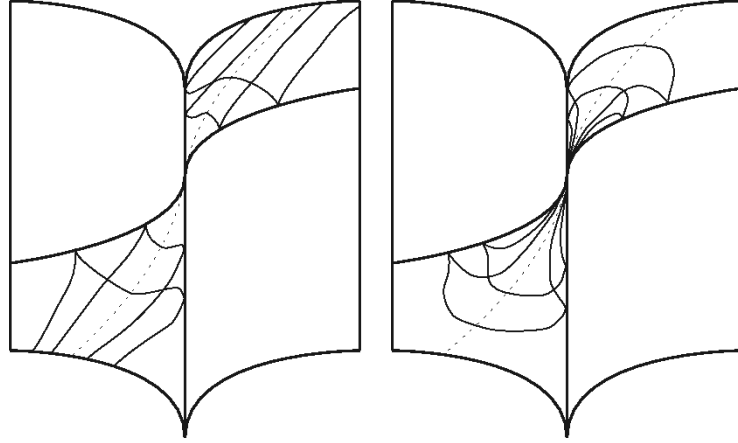


Figure 7: Integral curves of ω_{1s} and ω_{1n} on images of cuspidal edges.

for $k \geq 3$. With notation as before, consider

$$(a_{20}, a_{30}, \dots, b_{20}, b_{30}, \dots, b_{12}, \dots, b_{03}, \dots)$$

as a coordinate system of \mathcal{C}_k (cf. (1.2)). Define a subset of \mathcal{C}_k by

$$\begin{aligned} \mathcal{N} = & \{b_{20} = 0, b_{30} - a_{20}b_{12} = 0\} \cup \{b_{20} = 0, D_{asy} = 0\} \\ & \cup \{b_{20} = 0, 4b_{12}^3 - b_{03}^2b_{30} = 0\} \cup \{b_{20} = 0, D_{hmc} = 0\}. \end{aligned}$$

Then \mathcal{N} is an algebraic subset of \mathcal{C}_k of codimension 2. Since the singular set of a cuspidal edge is a curve, generically it will avoid the set \mathcal{N} . This implies that for generic cuspidal edges the configuration of $\omega_{lc}, \omega_{as}, \omega_{ch}$ are those in Propositions 3.1, 3.2, 3.4.

A Criteria for (3, 4) and (3, 4, 5)-cusp

In this section, we state criteria for (3, 4) and (3, 4, 5)-cusp. Set

$$c_1(t) = (t^3, t^4, 0, \dots, 0), \quad c_2(t) = (t^3, t^4, t^5, 0, \dots, 0),$$

and a map-germ $\gamma : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^n, 0)$, where $n \geq 2$ (respectively, $n \geq 3$) is called $(3, 4)$ -*cuspidal edge* (respectively, $(3, 4, 5)$ -*cuspidal edge*) if γ is \mathcal{A} -equivalent to the map-germ c_1 (respectively, c_2) at 0.

Proposition A.1. *A map-germ $\gamma : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^n, 0)$, where $n \geq 3$ (respectively, $n \geq 2$) is $(3, 4, 5)$ -*cuspidal edge* (respectively, $(3, 4)$ -*cuspidal edge*) if and only if*

- (i) $\gamma'(0) = \gamma''(0) = 0$,
- (ii) $\gamma^{(3)}(0), \gamma^{(4)}(0)$ and $\gamma^{(5)}(0)$ are linearly independent (respectively, linearly dependent, and $\gamma^{(3)}(0)$ and $\gamma^{(4)}(0)$ are linearly independent) where $(\)^{(i)} = d^i/dt^i$.

Although this proposition is known [4, 14], we give a sketch of proof for the readers who are not familiar with it.

Proof. Since

$$\Phi(\gamma(t))^{(i)} = d\Phi_0(\gamma^{(i)}(0)) \quad (i = 3, 4, 5)$$

holds for a map $\Phi : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ under the assumption $\gamma'(0) = \gamma''(0) = 0$, it is obvious that the conditions do not depend on the parameter and the coordinate system on \mathbf{R}^n .

To show the proposition, it is enough to show that $(t^3 + a_1t^5, t^4 + a_2t^5, a_3t^5, 0, \dots, 0) + O(6)$ is \mathcal{A} -equivalent to $(t^3, t^4, a_3t^5, 0, \dots, 0) + O(6)$, where $a_1, a_2, a_3 \in \mathbf{R}$. Considering the parameter change $t \mapsto t - a_2t^2/4 - (16a_1 + 3a_2^2)t^3/48$, it can be proved. \square

Using Proposition A.1, we have the following:

Proposition A.2. *Let $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$ be a cuspidal edge and $\gamma : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^2, 0)$ an ordinary cusp such that $df_0(\gamma''(0)) = 0$. Then $\hat{\gamma} = f \circ \gamma$ is a $(3, 4)$ -*cuspidal edge* at 0.*

Proof. Without loss of generality, one can assume that f is given by the form (1.2), and $\gamma(t) = (t^3 + a_4t^4 + a_5t^5 + O(6), t^2)$ ($a_4, a_5 \in \mathbf{R}$), because $df_0(\partial v) = 0$. Then $\hat{\gamma}(t) = (t^3 + a_4t^4 + a_5t^5, t^4/2, 0) + O(6)$. By Proposition A.1, we have the conclusion. \square

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